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IB Math Research Problem

The product of all factors of 2000 can be found using several methods. One of the methods I employed in the beginning is a primitive one—I wrote a computer program to calculate the product. The second method I used, which is the focus of this paper, provides me an answer that coincides with the answer given by the first method. The answer is 10240000000000 or, in scientific notation, 1.024×10^{33} or 20 to the power of 10.

Numbers	Factors						Product	Number of Factors	Log	Prime Factorization	Exponents of the Prime Factors
1	**1**						**1**	**1**	**ERROR**		
2	**1**	**2**					**2**	**2**	**1**	**2**	**1**
3	**1**	**3**					**3**	**2**	**1**	**3**	**1**
4	1	2	4				8	3	1.5	2^2	2
5	**1**	**5**					**5**	**2**	**1**	**5**	**1**
6	1	2	3	6			36	4	2	2x3	**3**
7	**1**	**7**					**7**	**2**	**1**	**7**	**1**
8	1	2	4	8			64	4	2	2^3	**3**
9	1	3	9				27	3	1.5	**3^2**	**2**
10	1	2	5	10			100	4	2	2x5	1 and 1
11	**1**	**11**					**11**	**2**	**1**	**11**	**1**
12	1	2	3	4	6	12	1728	6	3	2^2x3	2 and 1
13	**1**	**13**					**13**	**2**	**1**	13	**1**
14	1	2	7	14			196	4	2	2x7	1 and 1
15	1	3	5	15			225	4	2	3x5	**1 and 1**
16	1	2	4	8	16		1024	5	2.5	2^4	4
17	**1**	**17**					**17**	**2**	**1**	17	**1**
18	1	2	3	6	9	18	5832	6	3	2x3^2	1 and 2
19	**1**	**19**					**19**	**2**	**1**	19	**1**
20	1	2	4	5	10	20	8000	6	3	2^2x5	2 and 1

*__Prime Numbers__ are __BOLDED__

First I started with some simple sample cases and listed the first twenty positive integers and find out the their factors. I multiplied the factors together and put the product in the third column of my table. I also counted the number of factors each number has and put it in the fourth column. As suggested by the instructions, I calculated the (product – Original) and (Product / Original), but I could not find a pattern between the three columns. At last, I used logarithm to find out the relationship between number of factors of the original number and the product of the factors. I discovered that by dividing (log (product) / log (original number)) would provide me a set of numbers that have a interesting relation with the number of factors. As shown on the fifth column of the table, it is obvious that the log values are exactly half of the values of the number of factors. Since $\log_n(P)=X$ is the same as $P=N^x$, therefore $\log_n(\text{Product})=$ ½ the number of factors of N is the same as $N^{0.5 \text{ x Number of factors N has}}$. In other words, the product of all factors of a number is that number to the power of half of its number of factors. It can be better illustrated by the following general equation:

$$N^{0.5 \text{ x Number of factors N has}}$$

N: Any integers

This formula can be proven by previous numbers on the table. For example, 16 has five factors and the product of all of its factors should equal to $16^{0.5x5}$. Indeed, the product of all its factors is 1024.

I also think of an explanation that accounts for this special relationship between number of factors and products of all factors. For any integer, there is always an even number of factors (if duplicates are counted). The factors can be appropriately divided into groups of two. Interestingly enough, if the groups are properly divided, the product of any group would be the same as the original number. 16, for example, has factors: 1, 2, 4, 4, 8 and 16. 1 and 16, 2 and 8 and 4 and 4 all yield to 16. There will be three 16 produced. The product of those numbers would equal to 16^3. However, because 4 and 4 are duplicated, the product of the factor is only $16^{2.5}$. The products of the factors, without a doubt, would be the original number to the power of half the number of factors.

However, a new problem arises. The difficult part of the question is to actually find out how many factors do the number has. I found out that there is a relationship between the prime factorization of the number and the number of factors the number has. Initially, I did not realize the pattern because I did not write the prime factorization form in exponential form. However, as the value of my sample increases, I have to write them in the shorter form.

	Prime Factorization	Exponents of the Prime Factors	The Pattern	Number of Factors
1				**1**
2	**2**	**1**	1+1	**2**
3	**3**	**1**	1+1	**2**
4	2^2	2	2+1	3
5	**5**	**1**	1+1	**2**

6	2x3	**3**	3+1	4
7	**7**	**1**	1+1	**2**
8	2^3	**3**	3+1	4
9	**3^2**	**2**	2+1	3
10	2x5	1 and 1	(1+1)x(1+1)	4
11	**11**	**1**	1+1	**2**
12	2^2x3	2 and 1	(2+1)x(1+1)	6
13	13	**1**	1+1	**2**
14	2x7	1 and 1	(1+1)x(1+1)	4
15	3x5	**1 and 1**	(1+1)x(1+1)	4
16	2^4	4	4+1	5
17	17	**1**	1+1	**2**
18	2x3^2	1 and 2	(1+1)x(2+1)	6
19	19	**1**	1+1	**2**
20	2^2x5	2 and 1	(2+1)x(1+1)	6

I first looked closely at the prime factorization of prime numbers. I noticed that the number of factors is 2 but the prime factor only has a power of 1. For example, 19 has 2 factors: 1 and 19, but the prime factorization is 19^1. This initially suggests to me that the number of factors is double to the power of prime factorization. However, I quickly found a counterexample, 121(11^2). 121 has only three factors, but the exponent of its prime factor is 2. This prompts me the idea that the number of factors is one bigger than the exponent of the number's prime factor. I tried several other similar examples that contain only one prime factor, and discovered that I am correct. The general formula for number of factors of any number with only one prime factor is

1+(Exponent of the prime factor of **N**)

Unfortunately, 2000 has two prime factors: 2 and 5. As a result, I have to search for another smaller number that also has two prime factors. The easiest example is 10. According to my first rule, the number of factor it has should have a relationship with the exponents of its prime factors. The prime factors of 10 is 2^1 and 5^1 but 10 has 4 factors. From the confidence of my first rule, I believe there must be a relationship between (1+1), (1+1) and 4. Obvious enough, 2 times 2 equals 4. I further proved my hypothesis by choosing bigger numbers. 20 is the next number of my favorite list. The prime factorization of 20 is 2^2 and 5^1. According to my new second rule, 20 should have (2+1) x (1+1) factors. Sure enough, 20 have 6 factors. For integers that have more than one prime factors, the general formula should be:

1+(Exponent of the first prime factor of **N**) x 1+(Exponent of the second prime factor) x 1+(Exponent of the third prime factor) x 1+(Exponent of the forth), etc.

The relationship between prime factors and the number of factors a number has can be explained by sorting the factors into a table, as suggested by Mr. Morewood.

Since an integer has only one way to factorize it using prime numbers, and all numbers can be described using prime factorization according to a rule in Mathematics, any factors of a number can also be expressed as the product of the prime factors.

I first tried using the table method on 36:

1	**2**	**4**
3	**6**	**12**
9	**18**	**36**

I placed the square of the prime number and the prime number along the top and the left side of the table. Then I placed the other factors of the number inside the cells as shown. 6, for example, is the product of two times three. As a result, I placed 6 in the intersection between 2 and 3. I found out that because of the first left column and the top row, (2^0 and 3^0), there is one more column than powers of 2, and one more row than powers of 3. I tested this table on several other numbers, and I found out it works on all of the numbers containing two prime factors. By calculating the number of cells in the square, the number of factors of a number can be known. Because there is always an extra row and column, adding one to the powers and multiplying together would provide the answer essential to calculate the product of all factors of a number.

Before I tried the newly learnt method on a big number, such as 2000. I tried it on 100. The result is pleasing and I used the same method to calculate all the products of 2000 and verify my answer with the computer.

First I calculate the prime factorization of 2000.

2000 = 20 x 10

$20 = 2 \times 2 \times 5$

$10 = 2 \times 5$

$2000 = 20 \times 10 \times 10$

$2000 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 5$

or

$2000 = 2^4 \times 5^3$

According to my newly devised rule, 2000 should have (4+1) x (3+1) = 20 number of factors.

Knowing that 2000 has 20 factors, the product of all factors should equal:

$$N^{0.5 \text{ x Number of factors N has}}$$

$$= 20^{0.5x20}$$

$$= 20^{10}$$

$$= 10240000000000$$

The answer to the initial problem is 10240000000000.

For convenience, the two formulae mentioned above can be combined and form the following equation.

$$N^{0.5 \text{ x } (1+(\text{Exponent of the first prime factor of } \mathbf{N}) \text{ x } 1+(\text{Exponent of the second prime factor}) \text{ x } \ldots)}$$

(Please see attached for my rough work)