sin x=x-x^3/3!+x^5/5!-…
cos x=1-x^2/2!+x^4/4!-…
e^x = 1+x+x^2/2!+x^3/3!+…
lnx=(x-1)-$(x-1)^2$/2+$(x-1)^3$/3-…
ln(1+x)=x-(1/2)x^2+(1/3)x^3-…

Weighted Mean value theorem:

$$\int f(x)g(x) = f(c)\int g(x)$$

$g\in C[a,b]$ and $g'(x)\le k<1$ => converge to unique fix point from any point in [a,b]

$f\in C^2[a,b]$ and f(p)=0 and $f'(p)\neq 0$ => converge for some initial approx around p

$g\in C^3[a,b]$ and x=g(x) has sol'n p and $g'(p)\neq 1$ => converge for initial approx in [a,b] using Stevenson's method

Order of convergence

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^a}=\lambda$$

Aiken's method

$$\hat{p}=p_n-\frac{(\Delta p_n)^2}{\Delta^2 p_n}$$

Lagrange:

$$P(x)=\sum_{k=0}^{n} f(x_k)L_{n,k}(x),$$

$$L_{n,k}=\prod_{i=0,i\neq k}^{n}\frac{x-x_i}{x_k-x_i}$$

Neville's method

$$\frac{(x-x_j)P_{\text{no } j}(x)-(x-x_i)P_{\text{no } i}(x)}{(x_i-x_j)}$$

x_0	$P_0=Q_{0,0}$		
x_1	$P_1=Q_{1,0}$	$P_{0,1}=Q_{1,1}$	
x_2	$P_2=Q_{2,1}$	$P_{1,2}=Q_{2,1}$	$P_{0,1,2}=Q_{2,2}$

Cubic Spline

a) $S_j(x)$ is cubic on subinterval $[x_j,x_{j+1}]$ for each j=0,1,…n-1
b) $S(x_j)=f(x_j)$
c) $S_{j+1}(x_{j+1})=S_j(x_j+1)$
d) $S'_{j+1}(x_{j+1})=S'_j(x_j+1)$
e) $S''_{j+1}(x_{j+1})=S''_j(x_j+1)$
f) **Natural/Free:** $S''(x_0)=S''(x_n)=0$
Clamped: $S'(x_0)=f'(x_0)$ and $S'(x_n)=f'(x_n)$

Lagrange Error:

$$f(x)=P(x)+\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\ldots(x-x_n)$$

Proof: Consider $g(t)=f(t)-P(t)-[f(x)-P(x)]\prod_{i=0}^{n}\frac{t-x_i}{x-x_i}$,
which has n+2 roots => $\exists$ E s.t. $g^{(n+1)}(E)=0$

Divided difference

kth divided difference:

$$f[x_i,\ldots x_{i+k}]=\frac{f[x_{i+1},\ldots x_{i+k}]-f[x_i,\ldots x_{i+k-1}]}{x_{i+k}-x_i}$$

$$P_n(x)=f[x_0]+\sum_{k=1}^{n}f[x_0,x_i,\ldots,x_k](x-x_0)\ldots(x-x_{k-1})$$

$$=\sum_{k=0}^{n}s(s-1)\ldots(s-k+1)h^k f[x_0,x_i,\ldots,x_k]$$

x	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
		$f[x_0,x_1]=\frac{f[x_1]-f[x_0]}{x_1-x_0}$		
x_1	$f[x_1]$		$f[x_0,x_1,x_2]=\frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$	
		$f[x_1,x_2]=\frac{f[x_2]-f[x_1]}{x_2-x_1}$		$f[x_0,x_1,x_2,x_3]=\frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$
x_2	$f[x_2]$		$f[x_1,x_2,x_3]=\frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$	
		$f[x_2,x_3]=\frac{f[x_3]-f[x_2]}{x_3-x_2}$		$f[x_1,x_2,x_3,x_4]=\frac{f[x_2,x_3,x_4]-f[x_1,x_2,x_3]}{x_4-x_1}$
x_3	$f[x_3]$		$f[x_2,x_3,x_4]=\frac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2}$	
		$f[x_3,x_4]=\frac{f[x_4]-f[x_3]}{x_4-x_3}$		$f[x_2,x_3,x_4,x_5]=\frac{f[x_3,x_4,x_5]-f[x_2,x_3,x_4]}{x_5-x_2}$
x_4	$f[x_4]$		$f[x_3,x_4,x_5]=\frac{f[x_4,x_5]-f[x_3,x_4]}{x_5-x_3}$	
		$f[x_4,x_5]=\frac{f[x_5]-f[x_4]}{x_5-x_4}$		
x_5	$f[x_5]$			

Spectral Radius and Iterative Methods

(1) $\|A\|_2=[p(A^tA)]^{1/2}$ (2)p(A) $\le$ ||A|| for any ||.||
Following statements are equivalent:
(1)A is convergent matrix (2) $\|A^n\|=0$ as $n\to\infty$ for some ||.||
(3) p(A)<1 (4) $\|A^n\|=0$ as $n\to\infty$ for all ||.||
(5) A^n**x**=0 as as $n\to\infty$ for all **x**

Th'm: If p(T) <1, then $(1-T)^{-1}$ exists and $(1-T)^{-1}=1+T+T^2+\ldots$
Proof: (1) det(1-T)=0 => 1 is an eigenvalue
(2) $(I-T)S_m=1+T^2+..T^m=I-T^{m+1}$. Let $m\to\infty$.

Th'm: The sequence $\{x^{(k)}\}_{k=0}^{\infty}$ defined by $x^{(k)}=Tx^{(k-1)}+c$ converges to unique sol'n **x**=T**x**+**c** **iff** p(T)<1
Proof: (<=) $x^{(k)}=Tx^{(k-1)}+c=T(T(x^{(k-2)}+c)+c=\ldots=$ $=T^kx^{(0)}+(T^{(k-1)}+\ldots T+1)c$. $k\to\infty$ => $0+(1-T)^{-1}c$
(=>) For any **z**, construct $\mathbf{x}^{(0)}$**=x-z.** Assume $\{x^{(k)}\}$ converges to x. Note $x-x^{(k)}=(Tx+c)-(Tx^{(k-1)}+c)=T(x-x^{(k-1)})$.
∴ $\mathbf{x}-\mathbf{x}^{(k)}=T(x-x^{(k-1)})=T^2(x-x^{(k-2)})=T^k(x-x^{(0)})=T^kz$. As $k\to\infty$, LHS approaches 0. T^k**z**=0 for any **z**=> p(T)<1

Jacobi Iterative method: Write A=D-L-U
$T_j=D^{-1}(L+U)$ $c_j=D^{-1}\mathbf{b}$ $\mathbf{x}^{(k)}=T_j\mathbf{x}^{(k-1)}+c_j$
Gauss-Seidel: Write A=D-L-U
$T_g=(D-L)^{-1}U$ $c_g=(D-L)^{-1}\mathbf{b}$ $\mathbf{x}^{(k)}=T_j\mathbf{x}^{(k-1)}+c_j$

$$x_i^{(k)}=\frac{-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)})-\sum_{j=i+1}^{n}(a_{ij}x_j^{(k-1)})+b_i}{a_{ii}}$$

Error Bound: $\|\mathbf{x}-\mathbf{x}^{(k)}\|\le\|T\|^k\|x^{(0)}-x\|=(\|T\|^k/(1-\|T\|))\,\|x^{(1)}-x^{(0)}\|$

Definition of Residual Vector: $r=b-A\tilde{x}$

Strictly diag. dom.: $|a_{ii}|>\sum_{j=1,j\neq i}^{n}|a_{ij}|$

(1)Strictly diag. matrix is nonsingular.
(2)Gauss. elim. can be performed without row/column interchanges
(3)Computation will be stable with respect to round off error
(4)**Has LU factorization**

Positive Definite:

A is positive definite if it is
(1)symmetric **and**
(2)x^tAx<=>Ax·x > 0 for x $\neq 0$

A is positive definite matrix **then**
(1) A is nonsingular
(2) $a_{ii}>0$, for each i=1,2,…n
(3) $\max_{i\le k,j\le n}|a_{kj}|\le\max_{1\le i\le n}|a_{ii}|$
(4) $(a_{ij})^2<a_{ii}a_{jj}$ for each $i\neq j$
(5) All eigenvalues are positive
(6) GE solves without row-swap (All pivot element >0). Can write **A=LU**
Matrix A can be factored into LDL^t or LL^t iff A is positive definite

Leading Principle submatrix

$\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$ has 2, $\begin{bmatrix}2&-1\\-1&2\end{bmatrix}$ and itself

Th'm: A symmetrix matrix A is positive definite iff each of its leading principle submatrices has a positive determinant.

Permutation Matrix:

$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$A => $\begin{bmatrix}\text{1st row of A}\\\text{3rd row of A}\\\text{2nd row of A}\end{bmatrix}$

Properties of Vector Norm

(i) ||x||≥0 (ii)||x||=0 ⇔x=0
(iii)||ax||=|a|||x|| (iv)||x+y||≤||x||+||y||
$\|x\|_2$=Euclidean $\|x\|_\infty=\max|x_i|$
Th'm: The sequence of vectors $\{x^{(k)}\}$ converges to **x** with respect to $\|.\|_\infty$ iff $\lim_{k\to\infty}x_i^{(k)}=x_i$
Th'm: $\|x\|_\infty\le\|x\|_2\le\sqrt{n}\,\|x\|_\infty$

Matrix Norm $\|A\|=\max_{\|x\|=1}\|Ax\|$

(i) ||A||≥0 (ii)||A||=0 ⇔A=0
(iii)||aA||=|a|||A|| (iv)||AB||≤||A||||B||
(v)||A+B||≤||A||+||B||
$\|x\|_2$=Euclidean $\|x\|_\infty=\max|x_i|$

$$\|A\|_\infty=\max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|$$

Numerical Differentiation (unstable)

Forward diff.: $\frac{f(x_0+h)-f(x_0)}{h}-\frac{h}{2}f''(\xi)$ (Backward=h<0)

n+1-point formula: $x_0 \le \xi_1 \le x_0+2h$; $x_0-h \le \xi_2 \le x_0+h$

$$f'(x_j)=\sum_{k=0}^{n} f(x_k)L'_k(x_j)+\frac{f^{(n+1)}(\xi(x_j))}{(n+1)!}\prod_{k=0,k\neq j}^{n}(x_j-x_k)$$

3-point formula: (End point)

$$f'(x_0)=\frac{1}{2h}(-3f(x_0)+4f(x_0+h)-f(x_0+2h))+\frac{h^2}{3}f^{(3)}(\xi_1)$$

3-point formula: (Mid point):

$$f'(x_0)=\frac{1}{2h}(f(x_0+h)-f(x_0-h))+\frac{h^2}{6}f^{(3)}(\xi_2)$$

Mid-point with round-off, error term: $\frac{\varepsilon}{h}+\frac{h^2}{6}M$,

ε :round off bound for each step; M:3rd derivative bound

Derivation: Differentiating LaGrange with error term

Higher order derivatives: $x_0 \le \xi_3 \le x_0+2h$;

$$f''(x_0)=\frac{1}{h^2}(f(x_0-h)-2f(x_0)+f(x_0+h))+\frac{h^2}{12}f^{(4)}(\xi_3)$$

Derivation: Consider Taylors of $f(x_0+h)$ and $f(x_0-h)=f(x_0)-f'(x_0)h+(1/2)f''(x_0)h^2$…. Find linear combinations of those two such that the coefficient for $f'(x_0)$ is 0, Rearrange for $f''(x_0)$.

Numerical Integration (stable):

Trapezoid Rule: $\frac{h}{2}[f(x_0)+f(x_1)]-\frac{h^3}{12}f''(\xi$

Simpson's Rule: $\frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)]-\frac{h^5}{90}f^{(4)}(\xi)$

Def'n: Degree of accuracy / precision: largest n s.t. the approximation is exact for x^k, for $k \le n$

Composite Simpson's: $\frac{h^5}{90}\sum_{j=1}^{n/2}f^{(4)}(\xi_j)=\frac{b-a}{180}h^4 f^{(4)}(\xi)$.

Total round off error: $(b-a)\varepsilon$, ε is the bound for round-off.

Composite Trapezoid: $\frac{b-a}{12}h^2 f''(\xi)$

Adaptive Quadrature:

Theory

If h=(a+b)/2, then S(a,b) has error $(h^5/90)f^{(4)}(e_1)$, and $(S(a,(a+b)/2)+S((a+b)/2,b))$ has error $(1/16)(h^5/90)f^{(4)}(e_2)$

$$\left|\int_a^b f(x)dx-S(a,\frac{a+b}{2})-S(\frac{a+b}{2},b)\right|\approx\frac{1}{15}\left|(S(a,b)-S(a,\frac{a+b}{2})-S(\frac{a+b}{2},b))\right|$$

Procedure

Calculate S(a,b), S(a,(a+b)/2),S((a+b)/2,b).

If $\frac{1}{15}\left|(S(a,b)-S(a,\frac{a+b}{2})-S(\frac{a+b}{2},b))\right|$ > tolerance,

repeat procedure on the subinterval with tolerance=tolerance/2

Initial Value Problem:

f(t,y) satisfies **Lipschitz condition** in y on set $D \in R^2$ if a constant L>0 exists with $|f(t,y_1)-f(t,y_2)| \le L|y_1-y_2|$

With $\frac{dy}{dt}=f(t,y), 0\le t\le 2, y(a)=\alpha$, If f is continuous and satisfies Lipschitz in variable y, then the IVP is **well-posed** (unique sol'n exists and can be perturbed.).

Euler's Method: **Difference Equation:** $w_0=a$; $w_{i+1}=w_i+hf(t_i,w_i)$

Error: $|y(t_i)-w_i| \le \frac{hM}{2L}[e^{L(t_i-a)}-1]$

Difference Method: $w_0=a$; $w_{i+1}=w_i+\phi\, hf(t_i,w_i)$

Local Trunc. Err.:

$$T_{i+1}(h)=\frac{y_{i+1}-(y_i+h\phi(t_i,y_i))}{h}=\frac{y_{i+1}-y_i}{h}-\phi(t_i,y_i)$$

Euler local trunc. err: $T_{i+1}(h)=(h/2)y''(e)$ => O(h) for bounded y''

Higher order Taylor Methods:

$w_0=a$; $w_{i+1}=w_i+hT^{(n)}(t_i,w_i)$,

where $T^{(n)}(t_i,w_i)=f(t_i,w_i)+\frac{h}{2}f'(t_i,w_i)+\ldots+\frac{h^{n-1}}{n!}f^{(n-1)}(t_i,w_i)$

Finding y''(t): $f'=y''(t)=\frac{\partial f}{\partial t}(t,y(t))+\frac{\partial f}{\partial y}(t,y(t))\cdot f(t,y(t))$

Taylor Expansion of Two variables:

$$f(t+\alpha,y+\beta)=f(t,y)+\alpha\frac{\partial f}{\partial t}(t,y)+\beta\frac{\partial f}{\partial y}(t,y)+$$

$$+\frac{\alpha^2}{2}\frac{\partial^2 f}{\partial t^2}(\lambda,\gamma)+\alpha\beta\frac{\partial^2 f}{\partial t\partial y}(\lambda,\gamma)+\frac{\beta^2}{2}\frac{\partial^2 f}{\partial y^2}(\lambda,\gamma)$$

Midpoint Method O(h^2)

$w_0=a$; $w_{i+1}=w_i+h\cdot f(t_i+h/2, w_i+(h/2)\cdot f(t_i,w_i))$

Derivation: Matching coefficients of

$T^{(2)}(t_{i+1},w_{i+1})=f(t_i,w_i)+\frac{h}{2}\frac{\partial f}{\partial t}(t_i,w_i)+\frac{h}{2}\frac{\partial f}{\partial y}(t_i,w_i)\cdot f(t_i,w_i)$ with

$a_1 f(t+\alpha_1, y+\beta_1)$ => $a_1=1$; $a_1\alpha_1=h/2$; $a_1\beta_1=(h/2)f(t,y)$

Modified Euler Method O(h^2):

$w_0=a$; $w_{i+1}=w_i+\frac{h}{2}[f(t_i,w_i)+f(t_{i+1},w_i+hf(t_i,w_i))]$

Derivation: By matching $T^{(2)}$ with $a_1 f(t,y)+a_2 f(t+\alpha, y+\beta f(t,y)$

Gauss. Elimination O(n^3): $j=i+1\ldots n: m_{ji}=a_{ji}/a_{ii}; (E_j-m_{ji}E_i)\to E_j$;

*: $\sum_1^{n-1}(n-i)+(n-i)(n-i+1)$ +: $\sum_1^{n-1}(n-i)(n-i+1)$

Back substitution:

*: $1+\sum_1^{n-1}(n-i)+1=\frac{n^2+n}{2}$ +: $\sum_1^{n-1}(n-i-1)+1=\frac{n^2-n}{2}$

Partial pivot O(n^2): choose element from column with largest absolute value and interchange

Scaled partial pivoting O(n^2): Set s_i to be the element with largest absolute value in each row. When choosing a pivot, choose the element that has the largest absolute value when divided by the corresponding s_i in initial matrix.

Numerical Analysis Reference Sheet
(MACM 316, Simon Fraser University)

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Reference based on Numerical Analysis, by R.L. Burden and J.D. Faires, Brookes-Cole, Seventh Edition.