Vincent Chu
Block H
December 2000

# Investigating the Calculus of the Exponential Function

The function Ln(x) is defined as the slope of the exponential function, where $f(u) = x^u$ at the point (0,1). The limit definition of the derivative f'(u) is the limit of Ln(x).

The limit definition of $x^u$, where u is the variable, is:

The point is at (0,1). As a result, the value of u is 0. The above limit definition is thus simplified as follow:

This limit is approximated numerically for several values of x to find the graph of Ln(x). In other words, the base in the exponent is different each time a new x is used, and the slope of the graph at (0,1) is recorded. The slope of the graph is found by plugging in an arbitrary small number into the limit definition.

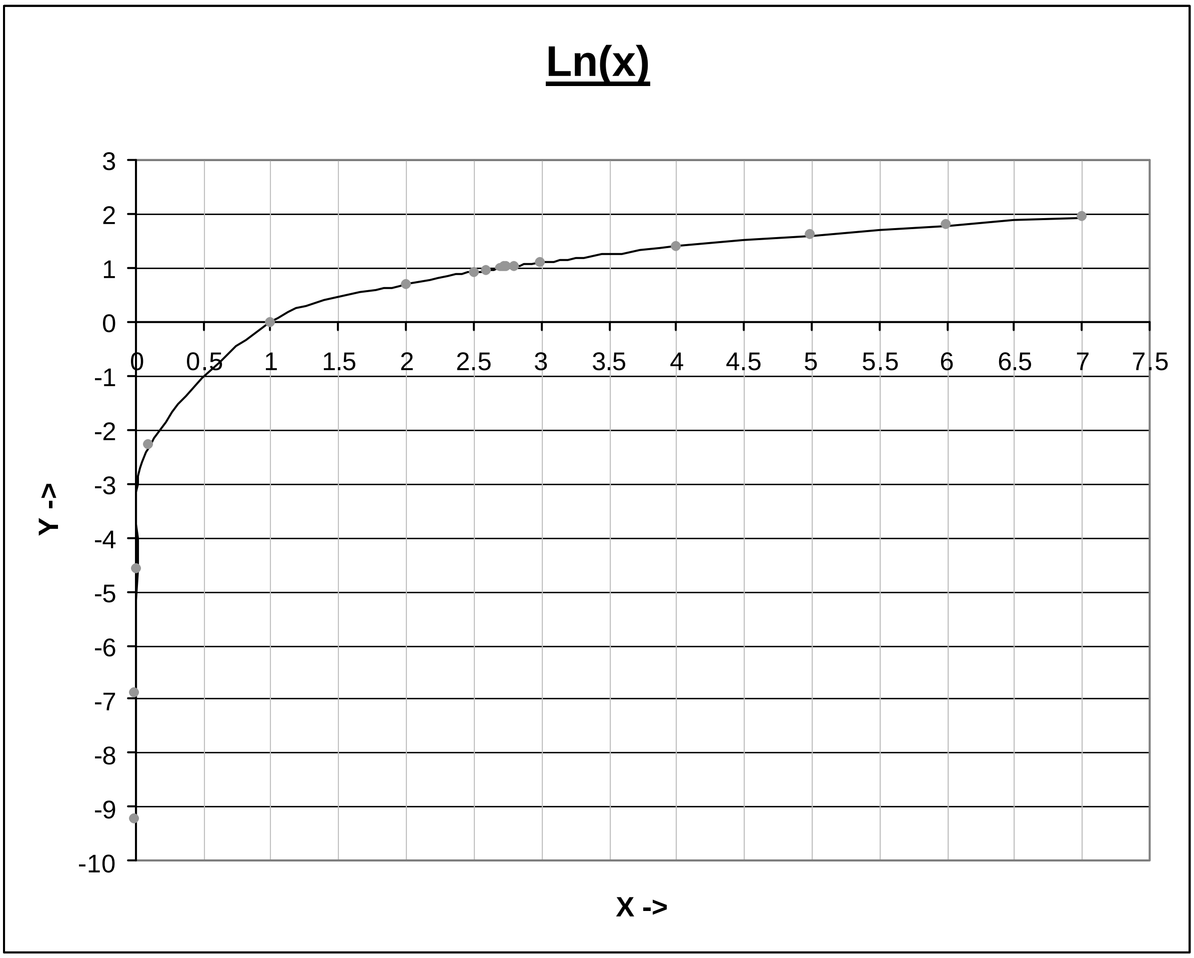
For example, if x is 2, the slope of the graph $x^u$ at (0,1) is:

Plugging in 0.0001, a randomly chosen small value, would give the approximate slope at that coordinate. The regular power rule does not work here. The value is recorded into the data table. Several other points are generated using the same method. A table of values for Ln(x) is created:

**Ln(x)**

| | |
|---|---|
| -1 | Imaginary |
| 0 | Undefined |
| 0.0001 | -9.2061 |
| 0.001 | -6.90537 |
| 0.01 | -4.60411 |
| 0.1 | -2.30232 |
| 1 | 0 |
| 2 | 0.693171 |
| 2.5 | 0.916333 |
| 2.6 | 0.955557 |
| 2.7 | 0.993301 |
| 2.7125 | 0.997921 |
| 2.725 | 1.002519 |
| 2.75 | 1.011652 |
| 2.8 | 1.029672 |
| 3 | 1.098673 |
| 4 | 1.38639 |
| 5 | 1.609567 |
| 6 | 1.79192 |
| 7 | 1.946099 |

The values can be plotted into a graph:




From the graph, it is shown that Y is also getting bigger as x is getting larger. The trend shows that y can increase forever if x is allowed to increase forever. However, when x is less than 1 and it gets closer and closer to zero, the y value becomes more and more negative. As x approaches zero, y approaches negative infinity. The range of the Ln(x) function is all real.

All the positive x values are "real". However, X cannot pass through zero, because zero is undefined. (Dividing by zero in the limit definition results in an undefined value). Ln(x), where $x<0$, would produce an imaginary answer. This is because a negative number to the power of a number between 0 and 1 would give an imaginary number. The domain of the Ln(x) function is: $x>0$

Although the function is increasing, the slope of the function is decreasing. It is evident from the graph that it becomes "flatter" as it goes up. The derivative formula for Ln(x) is $x^{-1}$. As the value of x increases, the slop decreases.

The graph can be used to approximate a solution to the equation Ln(x) = 1. More data points are calculated to try to make a more accurate approximation. From the values of the table, the solution should be between 2.7125 and 2.725. Using linear approximation, the solution should be at around x = 2.71875.

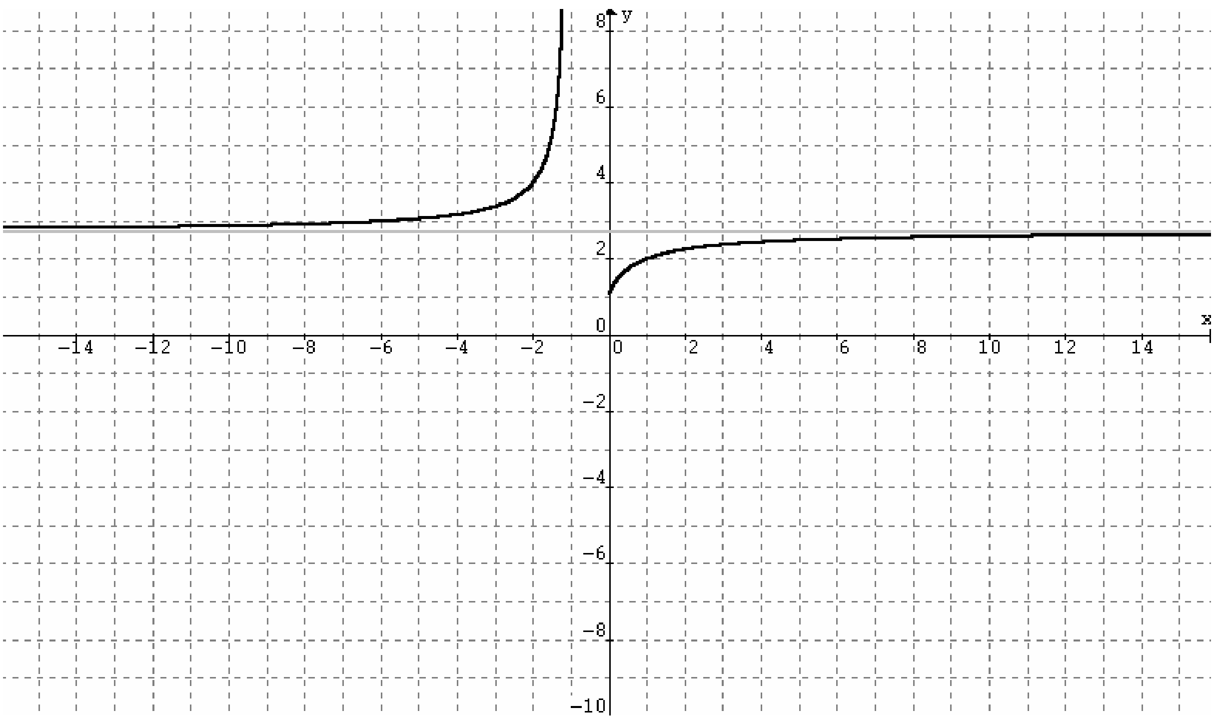
⋮

| | |
|---|---|
| 2.7125 | 0.997921 |
| 2.725 | 1.002519 |

⋮

Using the sequence, ,a better solution can be approximated. The sequence can be rearranged into the following:

The horizontal asymptote of the "sequence" is the solution to the equation Ln(x) = 1. If the sequence is made into a function, $a_n$ be the y value, and n be the x value, the following graph can be produced:



The limit of this graph as x approaches infinity is the solution to the equation Ln(x) = 1. The value can be approximated by plugging in arbitrary big numbers into the function:

| | |
|---|---|
| 1.00E+04 | 2.718145913 |
| 1.00E+06 | 2.718280469 |
| 1.00E+10 | 2.718282053 |
| Infinity | e |

The expected value of Euler's Number is 2.71828182845904523536028747135….. The number is named after Leonhard Euler who first introduced it in the 18th centaury. The Swiss mathematician lived in the years of 1707-1783, and has perfected many concepts and techniques in mathematics, physics and astronomy. He is the founder of pure mathematics, and developed "analysin infinitorum". He is remembered for the development of the theory of trigonometric and logarithmic functions, and the simplification of analytical operations in geometry. He might be called the Beethoven of Mathematics in that he has contributed in many areas of mathematics. He is responsible for the "discovery" of the Euler line and the barycenter of the triangle. Many notations are also invented by him, namely the f( ) notation, the naming of sides and angles in a triangle, and most importantly in this context, the use of the symbol e for the base of natural logarithms. Euler has advanced the use of infinitesimals and infinite quantities, and "e" is one of the most important numbers in mathematics.

Using the limit definition of Ln(x) to express Ln($x^n$), the power rule for Ln(x) can be found:

The power rule for $Ln(x^n)$ is $Ln(x^n) = n\ (Ln(x))$. It can be used to evaluate $Ln(e^x)$ to show that $Ln(e^x) = x$. Because an inverse function cancels out function, the fact that $Ln(e^x) = x$ shows $Ln(x)$ is the inverse of $e^x$:

Since the solution to the equation $Ln(x) = 1$ is e, $Ln(e)$ equals 1. $Ln(e^x)$ does equal x.

$Ln(x)$ is indeed the inverse of $e^x$. Conversely, $e^x$ is also the inverse of $Ln(x)$, and in $e^{Ln(x)}$, $e^x$ and $Ln(x)$ cancel each other out. $e^{Ln(x)}$ also equals x.

The limit definition can again be used to find the derivative of $e^x$.

As shown before, the limit definition of $Ln(x)$ is $(x^h-1)/h$. Here, $(e^{h-1})/h$ would thus equal $Ln(e)$.

Clearly, the derivative of $e^x$ is $e^x$. The limit definition of derivative can also be used to find the derivative of $a^x$, with "a" being any real number. However, the chain rule can also be used to find the derivative:

Based on the fact that an inverse function cancel out a function, the chain rule can be used to find a formula for the derivative of the Ln(x) function:

It is now known that the derivative of $e^x$ is $e^x$ and the derivative of Ln(x) is $x^{-1}$. The integral of $e^x$ would thus be $e^x$, and the integral of $x^{-1}$ is Ln(x):

Using the chain rule, the integral of Ln(x) can be found:

* The limit of the sequence as n goes to infinity is the solution to the equation Ln(x) = 1. As n goes to infinity, $a_n$ gets closer and closer to e, which is essentially the answer to the equation.

It is related to the equation Ln(x) = 1 in that it is basically the same as the limit definition of Ln(x) = 1, except for a small substitution of variable:

The given sequence merely have 1/n instead of h. As n approaches infinity, 1/n approaches zero. And that satisfies the original limit definition as $h \to 0$. As a result, the limit of Ln(x) =1 can be found by plugging in huge numbers into the sequence.